Research Statement

Jonathan Wells

1.1 Overview

My research lies in the intersection of Probability, Mathematical Physics, and Algebra, in the field of Random Matrix Theory. Currently I study the solvability of β-ensembles of random matrices for integer values of β beyond 1, 2 and 4.

Random matrix theory is the study of the eigenvalue statistics of ensembles of matrices, which are a collection of (typically) square matrices along with a probability measure defined on this set. Often, the probability measure is presented as the distribution of individual entries of the matrix, and then extended to the entire collection by specifying the inter-entry dependence. This probability measure then induces a probability measure on the eigenvalues of the matrix. Although a rich field of study on its own, random matrix theory also enjoys rather abundant application throughout mathematics and physics, since the eigenvalue statistics of many matrix ensembles can be used to model a wide variety of phenomena. The statistics of discrete energy levels in atomic spectra bear many of the same features as the eigenvalues of Hermitian matrices. Additionally, these same Hermitian ensembles can be used to describe the dynamics of a multi-particle system in one dimension interacting via a repulsive force and subject to a fixed potential. Alternatively, the *Wishart* ensemble of matrices can be used in the estimation of the covariance matrix for a population vector, given a large sample. The widespread applicability of random matrices evinces a universal paradigm—a collection of theorems akin to the classical Central Limit Theorem. But the utility of such theorems depends on an available supply of *solvable* ensembles in each universality class—collections of matrices for which the densities of eigenvalues can be expressed in terms of 'known' functions whose properties and asymptotics are well-studied.

The β-ensembles are one such collection, and are composed of random matrices whose eigenvalue densities take a common form, indexed by a non-negative, real parameter β. The classic β-ensembles ($\beta = 1, 2, 4$) correspond to Hermitian matrices with real, complex, or quaternionic Gaussian entries, and were first studied in the 1920s by John Wishart in multivariate statistics [17] and the 1950s by Eugene Wigner in nuclear physics [16]. In the subsequent decade, Freeman Dyson [7] unified a previously disparate collection of random matrix models by demonstrating that the three classic β-ensembles are each variations of a single action on random Hermitian matrices (representing the three associative division algebras over $\mathbb{R}$). More recently, the development [6] of matrix models representing arbitrary, non-negative values of β, as well as the discovery and expansion of Central Limit Theory-like results [12] lead to a renewed focus on these ensembles. In addition to their historical role in the development of random matrix theory, the classic β-ensembles remain essential to the current study of random matrices due to their membership in the class of *integrable* probability models—a somewhat nebulously-defined collection of objects which are enriched by some essential, overarching algebraic structure. The $\beta = 2$ ensemble is an example of a *determinantal point process*, while the $\beta = 1, 4$ ensembles are examples of *Pfaffian point processes.*

My current research uses tools from the exterior and shuffle algebras in order to unify the structure of the $\beta = 1, 2, 4$ ensembles, and to eventually represent β-ensembles as *Hyperpfaffian point processes*, when β is a square integer.

1.2 Background

1.2.1 β-Ensembles

In the sequel, suppose μ is a finite measure on $\mathbb{R}$ (historically, $d\mu(x) = \exp(-x^2/2)dx$). For each $\beta \in \mathbb{R}_+$, consider the N-point process specified by the joint probability density

$$\rho_N(x_1, \ldots, x_N) = \frac{1}{Z_N(\beta)N!} \prod_{i<j} |x_i - x_j|^\beta$$

where $Z_N(\beta)$ denotes the *partition function* of β, and is the normalizing constant required for ρ_N to be a probability density function. For each $1 \leq n \leq N$, define the nth correlation function by

$$R_n(x_1, \ldots, x_n) = \frac{1}{Z_N(\beta)(N-n)!} \int_{\mathbb{R}^{N-n}} \rho_N(x_1, \ldots, x_n, y_1, \ldots, y_{N-n})\, d^{N-n}\mu(y)$$

where $d^{N-n}\mu(y)$ denotes the $(N-n)$-fold product measure on $\mathbb{R}^{N-n}$. For β-ensembles, the correlation functions are nothing more than rescaled marginal density functions, and are completely determined by the joint density function. However, the study the local statistics of the ensemble is made simpler by using the correlation functions in place of the marginal densities.

When $\beta = 1$, 2, or 4, these correlation functions can be rewritten in a particularly nice form. For $\beta = 2$, elementary matrix operations and Fubini's Theorem can be used to show that

$$R_n(x_1, \ldots, x_n) = \frac{1}{Z_N(2)} \det\left(K(x_i, x_j)_{1 \leq i,j \leq n}\right)$$

where the *kernel* $K(x, y)$ is a certain square integrable function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that can most easily be expressed in terms of the orthogonal polynomials for the measure μ. The details of this derivation are given in [12].

And for $\beta = 1$ or 4,

$$R_n(x_1, \ldots, x_n) = \frac{1}{Z_N(\beta)} \mathrm{Pf}\left(K(x_i, x_j)_{1 \leq i,j \leq n}\right)$$

where $\mathrm{Pf}(A)$ denotes the Pfaffian of an antisymmetric matrix A (with $\mathrm{Pf}(A) := \sqrt{\det(A)}$) and where $K(x, y)$ is an antisymmetric matrix kernel. This result was first shown for Hermitian ensembles by Mehta in [11], and then for general weights by Mehta and Mahoux in [8] (except for the case $\beta = 1$ and N odd). Finally, the last remaining case was given by Adler, Forrester and Nagao in [10]

Of fundamental concern in the theory of random matrices is the behavior of eigenvalue statistics of matrix ensembles as $N \to \infty$. The immediate advantage of these determinantal and Pfaffian formulations for the correlation functions is that the fundamental characteristics of the eigenvalues are encoded in the kernel function, which does not increase in complexity as N grows large, considerably simplifying the asymptotic analysis of the β-ensemble.

1.2.2 The Method of Tracy and Widom

Derivations of the determinantal/Pfaffian forms of the correlation functions have been presented in numerous guises over the past several decades. However, of particular note is the method of Tracy and Widom [15], which evokes the underlying algebraic structure of the ensemble, and proceeds by first establishing that the partition function $Z_N(\beta)$ takes a determinant/Pfaffian form, and then uses the Sylvester Determinant Identity

$$\det(I_m + BA) = \det(I_n + AB) \qquad A \in M_{n\times m}(\mathbb{R}),\ B \in M_{m\times n}(\mathbb{R})$$

to show that the correlation functions must have the same form as well. For $\beta = 2$, the result follows almost immediately from the application of the Sylvester Identity, while the proof in the case when β is 1 or 4 required considerably more ingenuity.

This complication arises chiefly from the fact that, for most of the history of random matrix theory, results involving Pfaffians were often stated in terms of a *quaterion* determinant of a matrix, and calculations involving the Pfaffian were performed by first transforming the expressions into corresponding ones involving the determinant. However, the Pfaffian can also be viewed in an independent light as a particular evaluation in the exterior algebra of a vector space, and doing so allows for further generalization of the Tracy-Widom method.

1.2.3 The Exterior Algebra

For a vector space of finite even dimension $N = 2M$, the kth exterior power $\bigwedge^k V$ of V consists of all antisymmetric k-tensors of elements in V, and the exterior algebra $\bigwedge V$ is formed from the direct sum of all the exterior powers of V, with multiplication given by the antisymmetric tensor operation. Of key importance is the exterior square $\bigwedge^2 V$, whose elements bijectively correspond to $N \times N$ antisymmetric matrices. Under this correspondence, the Pfaffian of an antisymmetric matrix A can be obtained by taking the Mth power of the associated antisymmetric tensor. That is, if $\{v_1, \ldots, v_n\}$ is a basis for V and $\omega \in \bigwedge^2 V$ is the antisymmetric tensor associated with the matrix A, then

$$\frac{\omega^{\wedge M}}{M!} = (\mathrm{Pf} A)\, v_1 \wedge \cdots \wedge v_N$$

But from this perspective, Pfaffian identities naturally arise as structural properties of the exterior algebra. In particular, the Sylvester Determinant Identity has a Pfaffian analogue

$$\frac{\mathrm{Pf}(Z^{-1} + B^T AB)}{\mathrm{Pf} Z^{-1}} = \frac{\mathrm{Pf}(A^{-1} + BZB^T)}{\mathrm{Pf} A^{-1}} \qquad A \in M_n(\mathbb{R}),\ B \in M_{n\times m}(\mathbb{R}),\ Z \in M_m(\mathbb{R})$$

which then can be used to give a nearly identical proof of the Pfaffian form for the $\beta = 1, 4$ correlation functions as was used to prove the determinantal form for the $\beta = 2$ correlation functions.

1.2.4 Partition Functions as Pfaffians

For the classical β-ensembles, the first step for rewriting the correlation functions as determinants/Pfaffians is to observe that the partition function $Z_N(\beta)$ is itself a determinant/Pfaffian of a matrix of integrals of appropriately chosen orthogonal polynomials. One way to do so is to apply the Andreief determinant identity [1] to the partition function $Z_N(\beta)$. The result follows immediately when $\beta = 2$, and by viewing the Pfaffian as the square root of a determinant, the result can also be shown when $\beta = 1, 4$ with some additional finesse. But when the Pfaffian is viewed from the context of the exterior algebra, the Andreief determinant identity can be extended to analogous Pfaffian identities (referred to in the literature as the de Bruijn identities [5]). In fact, adopting this perspective illuminates an underlying algebraic structure, allowing the identity to be further generalized in the case when β is an arbitrary square integer.

Recalling the exterior algebra definition of the Pfaffian of an antisymmetric 2-tensor, we can define the *Hyperpfaffian*, $\mathrm{Pf}^{(\mathrm{L})}(\omega)$, of an antisymmetric L-tensor $\omega \in \bigwedge^L V$ by

$$\frac{\omega^{\wedge K}}{K!} = (\mathrm{Pf}^{(\mathrm{L})}\omega)\, v_1 \wedge \cdots \wedge v_N,$$

provided that V has dimension $N = KL$.

In 2002, Jean-Gabriel Luque and Jean-Ives Thibon [9] used techniques in the shuffle algebra to show that when $\beta = L^2$ is an even square integer, the partition function $Z_N(\beta)$ can be written as a Hyperpfaffian of an L form whose coefficients are integrals of Wronskians of suitable orthogonal polynomials. Then in 2011, Chris Sinclair [14] used combinatorial methods to show that the result also holds when β is an odd square integer.

In fact, as I show in a paper currently in preparation (and as outlined below) the shuffle algebra techniques first implemented by Thibon and Luque can, with some modification, be adopted to give a universal proof that $Z_N(\beta)$ can be written as a Hyperpfaffian when β is a square integer, regardless of whether β is even or odd.

1.3 Current Research

1.3.1 The Shuffle Algebra and de Bruijn's Identities

Given a set A and ring R, the shuffle algebra can be obtained by endowing the free R-algebra on A, $R\langle A\rangle$, with an additional product $\sqcup\!\sqcup$, which is first defined on basis elements and then extended linearly. For words $v = u_1 \dots u_k$ and $w = u_{k+1} \dots u_{k+n}$ in $R\langle A\rangle$, let

$$v \sqcup\!\sqcup w := \sum_{\sigma \in \mathrm{Sh}(k,n)} u_{\sigma^{-1}(1)} \dots u_{\sigma^{-1}(n+k)}$$

where $\mathrm{Sh}(k,n)$ is the subset of the symmetric group on $n+k$ letters consisting of permutations satisfying

$$\sigma(1) < \cdots < \sigma(k) \quad \sigma(k+1) < \cdots < \sigma(n).$$

That is, given two words v and w of length k and n, the product $v \sqcup\!\sqcup w$ is the sum of all $\binom{n+k}{k}$ words formed by interlacing the letters in v and w.

While the shuffle algebra is of great interest in its own right in representation theory and combinatorics (If V is a free R-module and V^* is its algebraic dual space, then $R\langle V^*\rangle_{\sqcup\!\sqcup}$ is isomorphic as a Hopf algebra to the graded dual of the tensor algebra $T(V)$ [13]), of more immediate relevance, it also appears to be the correct setting for performing the iterated integrals that appear in calculations of the partition function, $Z_N(\beta)$.

In particular, since the joint density function $\rho_N(x)$ for the β-ensemble is completely symmetric in the N variables, it can be restricted to the N-simplex $\Delta^N = \{x_1 < \cdots < x_N\}$ at the cost of a combinatorial factor $N!$. That is,

$$Z_N(\beta) = \int \prod_{i<j} |x_i - x_j|^\beta \, d^N\mu(x) = N! \int_{\Delta^N} \prod_{i<j} (x_i - x_j)^\beta d^N\mu(x).$$

This has the added benefit of eliminating a pesky absolute value sign in the case when β is an odd integer.

Now, take A to be a vector space of suitably integrable functions (say, $A = L^2(\mathbb{R}, \mu)$, where $d\mu(x) = e^{-x^2/2}\,dx$) and define a linear functional $\langle - \rangle$ on $R\langle A\rangle_{\sqcup\!\sqcup}$ by

$$\int_{\Delta^N} f_1(x_1)\cdots f_n(x_n)\,d\mu(x) \qquad \text{where } f_1\cdots f_n \text{ is a word in } R\langle A\rangle_{\sqcup\!\sqcup}\ .$$

Then $\langle - \rangle$ is, in fact, an algebra homomorphism, in the sense that

$$\Big\langle (f_1\cdots f_k) \sqcup\!\sqcup (f_{k+1}\cdots f_{k+n}) \Big\rangle = \Big\langle f_1\cdots f_k \Big\rangle \cdot \Big\langle f_{k+1}\cdots f_{k+n} \Big\rangle.$$

This result is equivalent to the celebrated lemma first proved by K.T. Chen in the context of cohomology of loop space [4].

With this lemma in hand, the de Bruijn identities for $\beta = L^2$ can now be recovered:

Theorem (Wells)**.** *Suppose N and L are positive integers, that $\mathbf{p} = \{p_1, p_2, \ldots, p_N; x\}$ is a complete family of monic orthogonal polynomials in the variable x, that $I = \{i_l\}$ and $J = \{j_k\}$ are subsets of $\{1, \ldots, N\}$ of size L, that $v_I = v_{i_1} \wedge \cdots \wedge v_{i_L}$, and that $\mathrm{Wr}(\mathbf{p}_I; x)$ is the Wronskian of a subset $\mathbf{p}_I = \{p_{i_1}, \ldots, p_{i_L}; x\}$. Then*

1. *if $\beta = L^2$ is even,*

$$Z_N(\beta) = \left\langle \prod_{i<j} (x_i - x_j)^\beta \right\rangle = \mathrm{Pf}^{(\mathrm{L})}\left(\Big\langle \mathrm{Wr}(\mathbf{p}_I; x) \Big\rangle\, v_I \right)$$

2. *if $\beta = L^2$ is odd and N is even,*

$$Z_N(\beta) = \left\langle \prod_{i<j} (x_i - x_j)^\beta \right\rangle = \mathrm{Pf}^{(\mathrm{L})}\left(\Big[\Big\langle \mathrm{Wr}(\mathbf{p}_I; x) \Big\rangle\, v_I \Big] \wedge \Big[\Big\langle \mathrm{Wr}(\mathbf{p}_J; y) \Big\rangle\, v_J \Big] \right)$$

The outline of this proof goes as follows: first, the integrand in $Z_N(\beta)$ is expressed as

$$Z_N(\beta) = \prod_{i<j} (x_i - x_j)^\beta = \mathrm{Pf}^{(\mathrm{L})}{}_{\sqcup\!\sqcup} \left(\mathrm{Wr}(p_{i_1}, \ldots, p_{i_L}) v_{i_1} \wedge \cdots \wedge v_{i_L} \right) \tag{1}$$

where $\mathrm{Pf}^{(\mathrm{L})}{}_{\sqcup\!\sqcup}$ denotes the Hyperpfaffian of an L-tensor with coefficients in the shuffle algebra $R\langle A\rangle_{\sqcup\!\sqcup}$. And second, the $\langle - \rangle$ homomorphism is applied to both sides of equation (1).

It is worth highlighting that the shuffle algebra proved its utility twice over (once in decomposing the Vandermonde power $\prod_{i<j}(x_i - x_j)^\beta$ as a Hyperpfaffian, and once again in facilitating the evaluation of the iterated integral of a product), evincing the fundamental algebraic structure of this problem.

1.3.2 Identities in the Exterior Algebra

Several algebraic identities of the determinant (the Laplace Expansion, the Cauchy-Binet Formula, and the Jacobi Minor Inverse Formula, for example) have been well-studied since the days of Jacobi, Cayley, and Sylvester. Given that the Pfaffian is the square root of the determinant, it is not terribly surprising that many of these identities have a Pfaffian analogue (a few of which appear in other guises elsewhere in the literature as the famed Wick Formulas [3],[2]). However, by representing an antisymmetric matrix as an antisymmetric

2-tensor, many of these Pfaffian identities can be realized simply as structural properties of the exterior algebra. And moreover, by adopting this perspective, it becomes clear that there is little unique about the Pfaffian of an antisymmetric 2-tensor—many of the same identities also hold for the Hyperpfaffian of an antisymmetric L-tensor.

Concise statements of the Laplace Expansion, the Cauchy-Binet formula, the Jacobi Minor Inverse formula, and the Sylvester identity are facilitated by introducing two auxillary transformations on the exterior algebra as follows:

For $\omega \in \bigwedge^L V$, define $\exp(\omega) \in \bigwedge V$ by

$$\exp(\omega) = \sum_{k=1} \frac{\omega^{\wedge k}}{k!}$$

and note that $\exp(\omega)$ is actually a finite sum, since $\omega^{\wedge k} = 0$ for all k with $kL > \dim V$. The *Hodge dual* operator $*$ on $\bigoplus_k \bigwedge^{2k} V$ is defined for $\alpha \in \bigwedge^{2k} V$ by $*(\alpha) = \beta$, where β is the unique antisymmetric $(n-2k)$-tensor so that

$$\alpha \wedge \beta = \frac{\mathrm{Pf}^{(2k)}(\alpha)}{(n/2k)!} v_1 \wedge \cdots \wedge v_n.$$

It turns out that in some cases, an antisymmetric $2k$-tensor α has a complementary antisymmetric $2k$-tensor α' with the property that

$$\exp(\alpha) = *[\exp(\alpha')],$$

in which case the coefficients of α contain essentially the same information as those of α'. Remarkably, if α is an antisymmetric 2-tensor with $\mathrm{Pf}(\alpha) \neq 0$ (corresponding to an invertible antisymmetric matrix A), then the complement α' always exists and corresponds to the matrix inverse of A.

Now, the Cauchy-Binet formula for Hyperpfaffians is equivalent to the observation that the k-homogeneous component of $\exp(\omega)$ is given by

$$\frac{\omega^{\wedge k}}{k!} = \sum_I \mathrm{Pf}^L(\omega_I)$$

where the sum is taken over all k-element subsets $I \subset \{1, 2, \ldots, N\}$, and where ω_I is called the I-minor of ω and denotes the L-formed obtained by setting the basis vectors corresponding to $i \notin I$ to 0.

Meanwhile, the Laplace Expansion for Hyperpfaffians arises by combining the Cauchy-Binet Formula with observation that if $\omega \in \bigwedge^L V$ and $\alpha \in \bigwedge^K V$ with $\omega = \alpha^{\wedge k}$, then $\mathrm{Pf}^{(L)}(\omega) = \mathrm{Pf}^{(K)}(\alpha)$.

By combining the equation $\exp(\alpha) = *[\exp(\alpha')]$ with the Cauchy-Binet formula, the Hyperpfaffians of minors of α can be expressed in terms of the Hyperpfaffians of minors of α'. In particular, when α is an antisymmetric 2-tensor, this result is equivalent to the Jacobi Minor Inverse Formula for Pfaffians.

In the language of the exterior algebra, the Pfaffian Sylvester Identity can be restated as

$$\frac{\mathrm{Pf}(\zeta' + B \cdot \alpha)}{\mathrm{Pf}(\zeta')} = \frac{\mathrm{Pf}(\alpha' + B^T \cdot \zeta)}{\mathrm{Pf}(\alpha')} \qquad \text{for } \alpha, \zeta \in \bigwedge^2 V \text{ and } B \in \mathrm{M}_n(\mathbb{R})$$

where $B \cdot \alpha$ denotes the antisymmetric 2-tensor given by

$$B \cdot \alpha = \sum_{i<j} \alpha_{ij}\, (Bv_i) \wedge (Bv_j) \qquad \text{when } \alpha = \sum_{i<j} \alpha_{ij}\, v_i \wedge v_j.$$

But more generally, by using the Hyperpfaffian in place of the Pfaffian, the same identity also holds for antisymmetric L-tensors α and ζ, provided that the complements of α and ζ exist.

1.4 Directions for Further Research

Recalling the method of Tracy and Widom, a streamlined proof of the Pfaffian formulation for the correlation functions of β-ensembles uses the Sylvester Identity rewrite the partition function $Z_N(\beta)$ as a generating function for the correlation functions with determinant/Pfaffian coefficients. But the existence of a Hyperpfaffian formulation for the partition function $\beta = L^2$, along with a Sylvester Identity for Hyperpfaffians, is highly suggestive of the existence of 'Hyperpfaffian' formulations for the $\beta = L^2$ correlation functions.

At present, the greatest obstacle to obtaining these Hyperpfaffian correlation functions is the observation that a generic antisymmetric L-tensor α need not have a complement α', and hence, Sylvester Identity might not apply to the the particular L-forms corresponding to the β-ensembles might not

But this may not be an insurmountable problem, since the antisymmetric L-tensors that arise from the β-ensemble model are highly non-generic. Indeed, several directions for further investigation are possible:

1. Since the the partition function $Z_N(\beta)$ depends only on the Hyperpfaffian of an L-tensor (and not the L-tensor itself), it is conceivable that the particular L-tensors arising from the β-ensemble can be replaced in a deterministic fashion with other L-tensors which do have complements.

2. Although the general β-ensemble may not readily admit a Hyperpfaffian representation, it is possible that the circular β-ensembles (which historically are more tractable than the classic Gaussian ensembles) give rise to L-forms which can be directly computed and chosen in such a way to allow application of the Sylvester identity.

3. Alternatively, there may be other point processes on $\mathbb{R}$ (either naturally arising or contrived for this purpose) whose associated L-tensors are reasonably well-behaved. It may then be possible to approximate the β-ensemble point processes using these 'Hyperpfaffian' models.

References

[1] C. Andreief. Note sur une relation pour les intégrales défines des produits des fonctions. *Memoires de la Societe (Royale) des Sciences*, 1883.

[2] F. A. Berezin. The method of second quantization. *Pure and Applied Physics*, 1966.

[3] E. R. Caianiello. Combinatorics and renormalization in quantum field theory. *Frontiers in Physics*, 1973.

[4] K. T. Chen. Iterated integrals and exponential homomorphisms. *Proceedings of the London Mathematical Society*, 1953.

[5] N. G. de Bruijn. On some multiple integrals involving determinants. *Journal of Indian Mathematical Society*, 1956.

[6] I. Dumitriu and A. Edelman. Matrix models for beta ensembles. *J. Math. Phys.*, 2002.

[7] F. P. Dyson. Statistical theory of the energy levels of complex systems. *Journal of Mathematical Physics*, 1962.

[8] M.L. Mehta G. Mahoux. A method of integration over matrix variables. *Journal de Physique I*, 1991.

[9] J. Luque and J. Thibon. Pfaffian and hafnian identities in shuffle algebras. *Advances in Applied Mathematics*, 2002.

[10] T. Nagao M. Adler, P.J. Forrester and P. van Moerbeke. Classical skew orthogonal polynomials and random matrices. *Journal of Statistical Physics*, 2000.

[11] M. L. Mehta. A note on correlations between eigenvalues of a random matrix. *Communications in Mathematical Physics*, 1971.

[12] M. L. Mehta. *Random Matrices*. Elsevier/Academic Press, 2004.

[13] C. Reutenauer. *Free Lie Algebras*. Oxford University Press, 1993.

[14] C. Sinclair. Ensemble averages when beta is a square integer. *Monatshefte fur Mathematik*, 2011.

[15] C. A. Tracy and H. Widom. Correlation functions, cluster functions, and spacing distributions for random matrices. *Journal of Statistical Physics*, 1988.

[16] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimension. *Annals of Mathematics*, 62, 1955.

[17] John Wishart. The generalized product moment distribution in samples from a normal multivariate population. *Biometrika*, 20 A, 1928.